Derivation of Panagopoulos' Equation[1]

The derivation was first given by:
D. J. Panagopoulos, N. Messini, A. Karabarbounis, A. L. Philippetis and L. H. Margaritis (2000). A Mechanism for Action of Oscillating Electric Fields on Cells. Biochemical and Biophysical Research Communications **272,** 634–640 (2000).

The following derivation is based on the notation of Wikipedia. We start with the equation of the damped and driven harmonic oscillator (https://en.wikipedia.org/wiki/Harmonic_oscillator):

Newton's second law takes the form

$$F(t) - kx - c\frac{\mathrm{d}x}{\mathrm{d}t} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}. \tag{1}$$

It is usually rewritten into the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\zeta\omega_0\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \frac{F(t)}{m}. \tag{2}$$

$\omega_0^2 = \frac{k}{m}$ is called the 'undamped angular frequency of the oscillator' and (3)

$\zeta = \frac{c}{2\sqrt{mk}}$ is called the 'damping ratio'. (4)

The parameters for Na^+- ions in a membrane channel are (see the papers of Panagopoulos et al.):
$m \cong 3.8 \cdot 10^{-26}$ kg, $k \cong 1.5 \cdot 10^{-24}$ kg/s^2, $c \cong 6.4 \cdot 10^{-12}$ kg/s
Therefore:

$$\omega_0^2 = \frac{k}{m} \cong 39.5\ \frac{1}{s^2} \text{ and } \omega_0 \cong 6.3\ \frac{1}{s} \tag{5}$$

$$\zeta = \frac{c}{2\sqrt{mk}} \cong 1.34 \cdot 10^{13} >> 1 \tag{6}$$

Because $\zeta >> 1$, we have an "*overdamped system* ($\zeta > 1$): The system returns to steady state without oscillating. Larger values of the damping ratio ζ return to equilibrium slower" (see Wikipedia).

To keep the connection to Wikipedia, we go to dimensionless variables and set:

$$\tau = \omega_0 \cdot t \text{ and } q(\tau) = x(t) \tag{7}$$

Equation (2) becomes:

$$\frac{d^2q}{d\tau^2} + 2\zeta\frac{dq}{d\tau} + q = \frac{F(\tau)}{\omega_0^2 m} \tag{8}$$

We consider the special case of a sinusoidal driving force:

$$F(t) = F_0 \cdot \sin(\omega t) \quad \text{or} \quad F(\tau) = F_0 \cdot \sin\left(\frac{\omega}{\omega_0}\tau\right) = F_0 \cdot \sin(\Omega\,\tau) \text{ with } \Omega = \frac{\omega}{\omega_0} \tag{9}$$

The general solution q is a sum of a transient solution q_t that depends on initial conditions, and a steady state solution q_{st} that is independent of initial conditions and depends only on the driving amplitude F_0, driving frequency ω, undamped angular frequency ω_0, and the damping ratio ζ:

$$q(\tau) = q_t(\tau) + q_{st}(\tau) \tag{10}$$

[1] Eq. (23) in: D. J. Panagopoulos, O. Johansson, G. L. Carlo (2015). Polarization: A Key Difference between Man-made and Natural Electromagnetic Fields, in regard to Biological Activity. *Scientific Reports* **5**, 14914; doi: 10.1038/srep14914

1. The transient solution $q_t(\tau)$:

According to Wikipedia the solution based on solving the ordinary differential equation is for $\zeta > 1$ and arbitrary constants c_1 and c_2:

$$q_t(\tau) = e^{-\zeta\tau}\left(c_1 e^{\tau\sqrt{\zeta^2-1}} + c_2 e^{-\tau\sqrt{\zeta^2-1}}\right) \tag{11}$$

Due to $\zeta >> 1$, which implies $\sqrt{\zeta^2 - 1} \cong \zeta$, and τ not too large (e.g. $\tau < 10^{-3}$ or $t < 1\ ms$, which is typical for the length of a pulse in mobile communication), we get the approximated function

$$q_t(\tau) \cong c_1 + c_2 e^{-2\zeta\tau} \tag{12}$$

And with eq. (3), (4) and (7) we have

$$x_t(t) \cong c_1 + c_2 e^{-2\zeta\omega_0 t} = c_1 + c_2 e^{-\frac{c}{m}t} \tag{13}$$

2. The steady-state solution $q_{st}(\tau)$:

Apply the "complex variables method" by solving the auxiliary equation (15) below for $Q_{st}(\tau)$ and then finding the real part of its solution ($q_{st}(\tau)$ = Re $Q_{st}(\tau)$): (14)

$$\frac{d^2 Q_{st}}{d\tau^2} + 2\zeta \frac{dQ_{st}}{d\tau} + Q_{st} = \frac{F_0}{\omega_0^2 m}(-i)\,[\cos(\Omega\tau) + i\sin(\Omega\tau)] = -i\,\frac{F_0}{\omega_0^2 m} e^{i\Omega\tau} \tag{15}$$

Supposing the solution is of the form

$$Q_{st}(\tau) = \frac{F_0}{\omega_0^2 m} \mathrm{A}\, e^{i(\Omega\tau + \phi)} \tag{16}$$

Inserting this solution into the differential equation gives:

$$-\Omega^2 A + 2\zeta i\,\Omega A + A = -i\,e^{-i\phi} = -i\cos\phi - \sin\phi \tag{17}$$

Equating the real and imaginary parts results in two independent equations

$$\left(\Omega^2 - 1\right) A = \sin\phi \quad \text{and} \quad -2\zeta\Omega A = \cos\phi \tag{18}$$

Therefore:

$$\tan\phi = -\frac{\Omega^2 - 1}{2\zeta\Omega} \tag{19}$$

We consider only the case, where $\Omega \geq 1$ or $\omega \geq \omega_0$. Therefore $\tan\phi \leq 0$ or $-\frac{\pi}{2} < \phi \leq 0$ and from equation (18) we get: A < 0.

Squaring both equations (18) and adding them together gives:

$$A^2[(\Omega^2 - 1)^2 + (2\zeta\Omega)^2] = 1 \tag{20}$$

Therefore

$$A = A(\zeta, \Omega) = -\frac{1}{\sqrt{(\Omega^2 - 1)^2 + (2\zeta\Omega)^2}} \tag{21}$$

The steady-state solution of eq. (15) is

$$Q_{st}(\tau) = -\frac{F_0}{\omega_0^2 m} \frac{e^{i(\Omega\tau + \phi)}}{\sqrt{(\Omega^2 - 1)^2 + (2\zeta\Omega)^2}} \quad \text{with} \quad \phi = -\arctan\left(\frac{\Omega^2 - 1}{2\zeta\Omega}\right) \tag{22}$$

With (6) and (9) eq. (22) can be written as:

$$Q_{st}(\tau) = -\frac{F_0}{\omega_0^2\, m\, 2\zeta\Omega}\,\frac{e^{i(\Omega\tau+\phi)}}{\sqrt{(\frac{\Omega^2-1}{2\zeta\Omega})^2+1}} = -\frac{F_0}{c\omega}\,\frac{e^{i(\Omega\tau+\phi)}}{\sqrt{(\frac{\Omega^2-1}{2\zeta\Omega})^2+1}} \tag{23}$$

Because $\zeta \cong 10^{13}$ (eq. (6)) and $1 \le \Omega \le 10^9$ ($10^8\,Hz$ ... $10^9\,Hz$ is the frequency-range of the mobile communication), we get

$$0 \le \frac{\Omega^2-1}{2\,\zeta\,\Omega} \le 10^{-4} \quad \text{and therefore} \quad \phi \cong 0 \;\text{ and }\; \sqrt{(\frac{\Omega^2-1}{2\zeta\Omega})^2+1} \cong 1.$$

This means: $Q_{st}(\tau)$ can be approximated with the function

$$Q_{st}(\tau) \cong -\frac{F_0}{c\omega}\, e^{i\,\Omega\tau} \tag{24}$$

The real part of this solution is:

$$q_{st}(\tau) = \text{Re}\; Q_{st}(\tau) \cong -\frac{F_0}{c\omega}\cos\Omega\tau \tag{25}$$

And with eq. (7) and (9) we have the approximation

$$x_{st}(t) \cong -\frac{F_0}{c\omega}\cos\omega t \tag{26}$$

3. The general solution $q(\tau) = x(t)$:

The general solution is the sum of eq. (11) and eq. (22). Because the system is extremely overdamped, we can us confine on the approximated sum (eq. (13), (26)). The result is:

$$x(t) = x_t(t) + x_{st}(t) \cong c_1 + c_2 e^{-\frac{c}{m}t} - \frac{F_0}{c\omega}\cos\omega t \tag{27}$$

4. Initial conditions

We consider the initial conditions

$$x(0) = 0 \quad \text{and} \quad \frac{dx}{dt}(t=0) = u_0 \qquad (28)\,/\,(29)$$

The first condition $x(0) = 0\,m$ results in

$$c_1 + c_2 = \frac{F_0}{c\omega} \tag{30}$$

The second condition results in

$$u_0 = -\frac{c}{m}\,c_2 \;\Rightarrow\; c_2 = -\frac{mu_0}{c} = -\frac{u_0}{2\zeta\omega_0} \tag{31}$$

Combining eq. (30) and (31) with (27) gives

$$x(t) \cong \frac{F_0}{c\omega}(1-\cos\omega t) + \frac{u_0}{2\zeta\omega_0}\left(1-e^{-2\zeta\omega_0 t}\right) \cong \frac{F_0}{c\omega}(1-\cos\omega t) + u_0 t + O(t^2) \tag{32}$$

For small t (t < $1\,ms$, which is typical for the length of a pulse in mobile communication) and small u_0 ($u_0 \cong 0\,\frac{m}{s}$ for Na^+- ions in a closed membrane channel) the term ~ u_0 can be ignored and we get the equation of Panagopoulos:

$$x(t) \cong \frac{F_0}{c\omega}(1-\cos\omega t) \tag{33}$$